

ON CARDINALITY BOUNDS INVOLVING THE WEAK LINDELÖF DEGREE

A. BELLA AND N. CARLSON

ABSTRACT. We give a general closing-off argument in Theorem 2.1 from which several corollaries follow, including (1) if X is a locally compact Hausdorff space then $|X| \leq 2^{wL(X)\psi(X)}$, and (2) if X is a locally compact power homogeneous Hausdorff space then $|X| \leq 2^{wL(X)t(X)}$. The first extends the well-known cardinality bound $2^{\psi(X)}$ for a compactum X in a new direction. As $|X| \leq 2^{wL(X)\chi(X)}$ for a normal space X [3], this enlarges the class of known Tychonoff spaces for which this bound holds. In 2.10 we give a short, direct proof of (1) that does not use 2.1. Yet 2.1 is broad enough to establish results much more general than (1), such as if X is a regular space with a π -base \mathcal{B} such that $|B| \leq 2^{wL(X)\chi(X)}$ for all $B \in \mathcal{B}$, then $|X| \leq 2^{wL(X)\chi(X)}$.

Separately, it is shown that if X is a regular space with a π -base whose elements have compact closure, then $|X| \leq 2^{wL(X)\psi(X)t(X)}$. This partially answers a question from [3] and gives a third, separate proof of (1). We also show that if X is a weakly Lindelöf, normal, sequential space with $\chi(X) \leq 2^{\aleph_0}$, then $|X| \leq 2^{\aleph_0}$.

Result (2) above is a new generalization of the cardinality bound $2^{t(X)}$ for a power homogeneous compactum X (Arhangel'skii, van Mill, and Ridderbos [2], De la Vega in the homogeneous case [9]). To this end we show that if $U \subseteq clD \subseteq X$, where X is power homogeneous and U is open, then $|U| \leq |D|^{\pi_\chi(X)}$. This is a strengthening of a result of Ridderbos [18].

1. INTRODUCTION.

The *weak Lindelöf degree* $wL(X)$ of a space X is the least cardinal κ such that for every open cover \mathcal{U} of X there exists $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ such that $X = cl(\bigcup \mathcal{V})$. In 1978 Bell, Ginsburg, and Woods [3] gave an example of a Hausdorff space X for which $|X| > 2^{wL(X)\chi(X)}$ and showed that if X is normal then $|X| \leq 2^{wL(X)\chi(X)}$. It was asked in Question 1 in [14]

2010 *Mathematics Subject Classification.* 54D20, 54D45, 54A25, 54E99.

Key words and phrases. cardinality bounds, cardinal invariants, locally compact, homogeneous.

The research that led to the present paper was partially supported by a grant of the group GNSAGA of INdAM.

whether the normality condition can be weakened to regular, yet to the authors' knowledge the question is even open under the Tychonoff assumption (see Question 2.14 below). Dow and Porter [10] showed that $|X| \leq 2^{\psi_c(X)}$ if X is H-closed, giving another class of spaces for which $|X| \leq 2^{wL(X)\chi(X)}$. As $|X| \leq 2^{L(X)\chi(X)}$ [1] and $|X| \leq 2^{c(X)\chi(X)}$ [13] for every Hausdorff space, it also follows trivially that if X is Lindelöf or X has countable chain condition then $|X| \leq 2^{wL(X)\chi(X)}$. More recently, Gotchev [12] has shown that if X has a regular G_δ -diagonal then the cardinality of X has the even stronger bound $wL(X)\chi(X)$.

We then have five classes of Hausdorff spaces X for which $|X| \leq 2^{wL(X)\chi(X)}$: Lindelöf, c.c.c., normal, H-closed, and spaces having a regular G_δ -diagonal. All of these properties can be thought of as “strong” in some sense. One may expect that these properties would be strong, as the cardinal invariant $wL(X)$ is generally regarded as a substantial weakening of the Lindelöf degree $L(X)$, to such an extent that even if X is normal it is not guaranteed that $wL(X) = L(X)$. (However, it is straightforward to see that if X is paracompact then $wL(X) = L(X)$, as pointed out in 4.3 in [3]). Also, $wL(X)$ is small enough of an invariant so that $wL(X) \leq c(X)$, where $c(X)$ is the cellularity of X , itself regarded as being “small”.

In §2 of this note we give other classes of spaces X for which $|X| \leq 2^{wL(X)\chi(X)}$. It is shown in Corollary 2.5 that if X is either Urysohn or quasiregular and has a dense subset D such that each $d \in D$ has a closed neighborhood that is H -closed, normal, Lindelöf, or c.c.c., then $|X| \leq 2^{wL(X)\chi(X)}$. The fundamental technique we develop is a closing-off argument given in the proof of Theorem 2.1. This theorem is a modified version of Theorem 2.5(b) in [6]. The latter theorem assumes a space with a dense set of isolated points, which we generalize in Theorem 2.1 to spaces with an open π -base \mathcal{B} such that $|B|$ is still “small” for each $B \in \mathcal{B}$; that is, $|B| \leq 2^\kappa$ for a suitably defined cardinal κ .

As every locally compact space is of pointwise countable type and thus $\chi(X) = \psi(X)$, it follows from Corollary 2.5 that the cardinality of a locally compact space is at most $2^{wL(X)\psi(X)}$ (Corollary 2.11). This result then generalizes the well-known bound $2^{\psi(X)}$ for a compactum X in a new direction. It is clear that $2^{\psi(X)}$ is not itself a cardinality bound for all locally compact spaces, as witnessed by a discrete space of size 2^c .

In Theorem 2.10 we give a short, direct proof that $2^{wL(X)\chi(X)}$ is a cardinality bound for any Hausdorff space X that is locally H-closed, regular and locally normal, locally Lindelöf, or locally c.c.c. (Note,

however, that this also follows from Corollary 2.5 in the case where X is Urysohn or quasiregular). It follows (again) that locally compact spaces X satisfy $|X| \leq 2^{wL(X)\psi(X)}$. The result that every regular, locally normal space X satisfies $2^{wL(X)\chi(X)}$ is a proper improvement of the Bell, Ginsburg, and Woods result for normal spaces.

In Theorem 2.7 we show that if X is a regular space with a π -base whose elements have compact closure, then $|X| \leq 2^{wL(X)\psi(X)t(X)}$. The proof does not use the main Theorem 2.1 and gives a third proof that the cardinality of a locally compact space X satisfies $|X| \leq 2^{wL(X)\psi(X)}$. It also give a partial answer to Question 4.1 in [3].

Also in §2 we show that if X is a weakly Lindelöf, normal, sequential space satisfying $\chi(X) \leq 2^{\aleph_0}$, then $|X| \leq 2^{\aleph_0}$. One might think of this as an analogue of Arhangel'skiĭ's result that if X is a Lindelöf, Hausdorff, sequential space with $\psi(X) \leq 2^{\aleph_0}$ then $|X| \leq 2^{\aleph_0}$.

The main closing-off argument given in Theorem 2.1 also has implications for the cardinality of spaces with homogeneity-like properties, which we give in §3. Recall a space X is *homogeneous* if for all $x, y \in X$ there exists a homeomorphism $h : X \rightarrow X$ such that $h(x) = y$, and X is *power homogeneous* if there exists a cardinal κ such that X^κ is homogeneous. It is shown in Corollary 3.7 that if X is locally compact and power homogeneous then $|X| \leq 2^{wL(X)t(X)}$. This represents an extension of the cardinality bound $2^{t(X)}$ for a power homogeneous compactum X given by Arhangel'skiĭ, van Mill, and Ridderbos [2] in a new direction. (De la Vega first established the bound $2^{t(X)}$ for compact homogeneous spaces in [9]). A key ingredient is Lemma 3.4, which slightly improves Ridderbos' bound $d(X)^{\pi\chi(X)}$ [18] for the cardinality of a power homogeneous space X and uses modified techniques from that paper. The cardinality bound $2^{wL(X)t(X)}$ for locally compact, power homogeneous spaces is then a “companion bound” to the bound $2^{wL(X)\psi(X)}$ for general locally compact spaces, both fundamentally proved with the same closing-off argument given in Theorem 2.1. Such companion bounds for power homogeneous spaces also appear in [6], [8], and [7].

For definitions of cardinal invariants and other notions not defined in this note, we refer the reader to [11], [15], and [16]. *All spaces are assumed to be Hausdorff.*

2. A CLOSING-OFF ARGUMENT INVOLVING THE WEAK-LINDELÖF DEGREE.

In this section we aim towards demonstrating that minor separation requirements on a space X (Urysohn or quasiregular) and the existence

of a π -base \mathcal{B} for X such that $|B| \leq 2^{wL(X)\chi(X)}$ for all $B \in \mathcal{B}$ are sufficient to guarantee that $|X| \leq 2^{wL(X)\chi(X)}$ (Corollary 2.3).

In Theorem 2.1 below we give the main closing-off argument at the core of our cardinality bound results. This theorem is similar to Theorem 2.5(b) in [6], but rather than the requirement that X has a dense set of isolated points, we make the weaker assumption that X has a π -base with “small” elements.

Recall that for a space X , the θ -closure of a subset $A \subseteq X$ is $cl_\theta A = \{x \in X : clU \cap A \neq \emptyset \text{ for all open sets } U \text{ containing } x\}$. A subset $D \subseteq X$ is θ -dense if $cl_\theta D = X$. The θ -density of space X is $d_\theta(X) = \min\{|D| : D \text{ is } \theta\text{-dense in } X\}$. It is straightforward to see that if X is regular then $cl_\theta A = clA$ for all $A \subseteq X$, and if X is *quasiregular*; that is, every non-empty open set contains a non-empty regular-closed set, then $d_\theta(X) = d(X)$.

Theorem 2.1. *Let X be a space and κ a cardinal such that $wL(X)t(X) \leq \kappa$. Suppose X has an open π -base \mathcal{B} such that $|B| \leq 2^\kappa$ for all $B \in \mathcal{B}$. Let \mathcal{C} be a cover of X consisting of compact subsets C of X such that $\chi(C, X) \leq \kappa$. Then there exists a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $X = cl_\theta(\bigcup \mathcal{C}')$ and $|\mathcal{C}'| \leq 2^\kappa$.*

Proof. For every $C \in \mathcal{C}$, we fix a collection \mathcal{U}_C of open subsets of X that forms a neighborhood base at C such that $|\mathcal{U}_C| \leq \kappa$. If $\mathcal{C}' \subseteq \mathcal{C}$, then define $\mathcal{U}(\mathcal{C}') = \bigcup \{\mathcal{U}_C : C \in \mathcal{C}'\}$. We note that each $C \in \mathcal{C}$ is a G_κ^c -set, as defined in Definition 3.3 in [8]. G_κ^c -sets are also referred to as regular G_κ -sets.

By induction we build an increasing sequence $\{A_\alpha : \alpha < \kappa^+\}$ of open subsets of X and an increasing chain $\{\mathcal{C}_\alpha : \alpha < \kappa^+\}$ of subsets of \mathcal{C} such that

- (1) $|\mathcal{C}_\alpha| \leq 2^\kappa$ and $|A_\alpha| \leq 2^\kappa$.
- (2) \mathcal{C}_α covers clA_α ,
- (3) if $\mathcal{V} \in [\mathcal{U}(\mathcal{C}_\alpha)]^{\leq \kappa}$ is such that $X \setminus cl(\bigcup \mathcal{V}) \neq \emptyset$, then $A_{\alpha+1} \setminus cl(\bigcup \mathcal{V}) \neq \emptyset$.

For limit ordinals $\beta < \kappa^+$, we let $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$. Then $|A_\beta| \leq 2^\kappa$ and hence $d(clA_\beta) \leq 2^\kappa$. By Lemma 3.5 in [8] we obtain a collection \mathcal{C}_β with the properties needed in (1) and (2).

Consider a successor ordinal $\beta + 1$. As \mathcal{B} is a π -base, for each $\mathcal{V} \in [\mathcal{U}(\mathcal{C}_\beta)]^{\leq \kappa}$ for which $X \setminus cl(\bigcup \mathcal{V}) \neq \emptyset$, there exists $B_\mathcal{V} \in \mathcal{B}$ such that $B_\mathcal{V} \subseteq X \setminus cl(\bigcup \mathcal{V}) \neq \emptyset$ and $|B_\mathcal{V}| \leq 2^\kappa$. Define

$$A_{\beta+1} = A_\beta \cup \bigcup \left\{ B_\mathcal{V} : \mathcal{V} \in [\mathcal{U}(\mathcal{C}_\beta)]^{\leq \kappa} \text{ and } X \setminus cl\left(\bigcup \mathcal{V}\right) \neq \emptyset \right\}$$

Observe that A_α is open for all $\alpha < \kappa^+$. As $|A_\beta| \leq 2^\kappa$, $|\mathcal{U}(\mathcal{C}_\beta)|^{\leq \kappa} \leq 2^\kappa$, and each $|B_\gamma| \leq 2^\kappa$, we have $|A_{\beta+1}| \leq 2^\kappa$ and thus $d(cl A_{\beta+1}) \leq 2^\kappa$. We again use Lemma 3.5 in [8] to obtain $\mathcal{C}_{\beta+1}$.

Let $\mathcal{C}' = \bigcup_{\alpha < \kappa^+} \mathcal{C}_\alpha$ and $F = \bigcup_{\alpha < \kappa^+} cl A_\alpha$. Note that $|\mathcal{C}'| \leq 2^\kappa$, F is closed since $t(X) \leq \kappa$, $F = cl(\bigcup_{\alpha < \kappa^+} A_\alpha)$, and \mathcal{C}' covers F . We now show that $X \subseteq cl_\theta(\bigcup \mathcal{C}')$. Suppose not and pick a point $x \in X \setminus cl_\theta(\bigcup \mathcal{C}')$. Then there exists an open set W containing x such that $\bigcup \mathcal{C}' \subseteq X \setminus cl W$. Hence for each $C \in \mathcal{C}'$, we see that $C \subseteq X \setminus cl W$ and since \mathcal{U}_C forms a neighborhood base at C there exists $U_C \in \mathcal{U}_C$ such that $C \subseteq U_C \subseteq X \setminus cl W$. Then, as \mathcal{C}' covers F , $\mathcal{U} = \{U_C : C \in \mathcal{C}'\}$ is an open cover of F . As $wL(X) \leq \kappa$ and F is regular-closed, it follows that $wL(F, X) \leq \kappa$ as $wL(X)$ is hereditary on regular-closed sets. We may find $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ such that $F \subseteq cl(\bigcup \mathcal{V})$. Since $W \cap \bigcup \{V : V \in \mathcal{V}\} = \emptyset$, we see that $x \in X \setminus cl(\bigcup \mathcal{V})$. As $\mathcal{V} \in [\mathcal{U}(\mathcal{C}_\alpha)]^{\leq \kappa}$ for some $\alpha < \kappa^+$, by condition (3) we have $A_{\alpha+1} \setminus cl(\bigcup \mathcal{V})$. But this is a contradiction since $A_{\alpha+1} \subseteq F \subseteq cl(\bigcup \mathcal{V})$. Thus, $X \subseteq cl_\theta(\bigcup \mathcal{C}')$. \square

We observe that in the above proof a chain $\{A_\alpha : \alpha < \kappa^+\}$ of open sets is inductively constructed by adding on members of a π -base. The closure of the union of the chain is then a regular-closed set F , and it follows that $wL(F, X) \leq wL(X)$. This procedure generalizes the case where X has a dense set of isolated points given in Theorem 2.5(b) in [6].

The following corollary gives a bound for the θ -density $d_\theta(X)$ of any space X with a π -base consisting of “small” elements.

Corollary 2.2. *Let X be a space with an open π -base \mathcal{B} such that $|B| \leq 2^{wL(X)\chi(X)}$ for all $B \in \mathcal{B}$. Then $d_\theta(X) \leq 2^{wL(X)\chi(X)}$.*

Proof. Let $\kappa = wL(X)\chi(X)$. We note that $\mathcal{C} = \{\{x\} : x \in X\}$ is a collection of compact subsets of X with character at most κ . By Theorem 2.1, there exists a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $X = cl_\theta(\bigcup \mathcal{C}')$ and $|\mathcal{C}'| \leq 2^\kappa$. Then, as \mathcal{C}' consists of singletons, we have $d_\theta(X) \leq |\bigcup \mathcal{C}'| = |\mathcal{C}'| \leq 2^\kappa$. \square

Corollary 2.3. *Let X be Urysohn or quasiregular, and suppose X has an open π -base \mathcal{B} such that $|B| \leq 2^{wL(X)\chi(X)}$ for all $B \in \mathcal{B}$. Then $|X| \leq 2^{wL(X)\chi(X)}$.*

Proof. If X is Urysohn it was shown in [4] that $|X| \leq d_\theta(X)^{\chi(X)}$. Thus, by Corollary 2.2,

$$|X| \leq d_\theta(X)^{\chi(X)} \leq (2^{wL(X)\chi(X)})^{\chi(X)} = 2^{wL(X)\chi(X)}.$$

If X is quasiregular then $d_\theta(X) = d(X)$. As $|X| \leq d(X)^{\chi(X)}$ for any Hausdorff space, we have $|X| \leq d_\theta(X)^{\chi(X)}$ and the result follows again from Cor. 2.2. \square

The space Z in Example 2.3 in [3] demonstrates that in Corollary 2.3 the condition that X is Urysohn or quasiregular is necessary. We give a brief description of the space Z . Let κ be an arbitrary uncountable cardinal and let A be any countable dense subset of \mathbb{P} . Let Z be the set $(\mathbb{Q} \times \kappa) \cup A$. If $q \in \mathbb{Q}$ and $\alpha < \kappa$, then a neighborhood base at (q, α) is $\{U_n(q, \alpha) : n \in \mathbb{N}\}$ where $U_n(q, \alpha) = \{(r, \alpha) : r \in \mathbb{Q} \text{ and } |r - q| < 1/n\}$. If $\alpha \in A$, a neighborhood base at α is $\{\{b \in A : |b - \alpha| < 1/n\} \cup \{(q, \alpha) : \alpha < \kappa \text{ and } |q - \alpha| < 1/n\} : n \in \mathbb{N}\}$. Z is an example of a weakly Lindelöf, first countable Hausdorff space with arbitrary cardinality κ . Furthermore, Z has a π -base of countable sets, namely the sets $U_n(q, \alpha)$ for $q \in \mathbb{Q}$ and $\alpha < \kappa$. Thus, in Corollary 2.3, the condition that X is Urysohn or quasiregular cannot be weakened to Hausdorff.

Corollary 2.4. *Let X be a space and \mathcal{B} an open π -base. Suppose for all $B \in \mathcal{B}$ that clB is H-closed, normal, Lindelöf, or has the countable chain condition. Then $d_\theta(X) \leq 2^{wL(X)\chi(X)}$ and if X is quasiregular or Urysohn then $|X| \leq 2^{wL(X)\chi(X)}$.*

Proof. Let $\kappa = wL(X)\chi(X)$ and let $B \in \mathcal{B}$. If clB is H-closed, then $|clB| \leq 2^{\psi_c(clB)} \leq 2^{\psi_c(X)} \leq 2^\kappa$ by Corollary 2.3 in [10]. If clB is normal, then $|clB| \leq 2^{wL(clB)\chi(clB)} \leq 2^\kappa$ by Theorem 2.1 in [3] and that fact that $wL(X)$ is hereditary on regular-closed sets. If clB is Lindelöf then $|clB| \leq 2^{L(clB)\chi(clB)} = 2^{\chi(clB)} \leq 2^{\chi(X)} \leq 2^\kappa$. If clB has the countable chain condition, then $|clB| \leq 2^{c(clB)\chi(clB)} \leq 2^{\chi(X)} \leq 2^\kappa$. In all cases we see that $|B| \leq |clB| \leq 2^\kappa$. The results now follow from Corollaries 2.2 and 2.3. \square

Corollary 2.5. *Let X be a space that is either quasiregular or Urysohn. Suppose there exists a dense subset $D \subseteq X$ such that each $d \in D$ has a closed neighborhood that is H-closed, normal, or Lindelöf, or c.c.c. Then $|X| \leq 2^{wL(X)\chi(X)}$.*

Proof. We show that X has a π -base with the property described in Cor. 2.4. Let U be a non-empty open set in X . There exists $d \in U \cap D$ and an open set V containing d such that clV is H-closed, normal, Lindelöf, or c.c.c. Let $B = U \cap V$. Then $d \in B$. As the H-closed and c.c.c. properties are hereditary on regular-closed sets, and normality and Lindelöf-ness are closed hereditary, we see that clB is H-closed, normal, Lindelöf, or c.c.c. As $B \subseteq U$, this shows X has a π -base with the properties described in Cor. 2.4. Thus, $|X| \leq 2^{wL(X)\chi(X)}$. \square

The following result of Dow and Porter [10] follows immediately, as any space with a dense set of isolated points is quasiregular.

Corollary 2.6 (Dow-Porter). *If X has a dense set of isolated points then $|X| \leq 2^{wL(X)\chi(X)}$.*

We show in Theorem 2.7 below that if X is regular with a π -base whose elements have compact closure, then $|X| \leq 2^{wL(X)\psi(X)t(X)}$. One can view this theorem as a variation of Corollary 2.4 above, where the hypotheses are strengthened resulting in a stronger cardinality bound. Note that the proof of Theorem 2.7 does not use Theorem 2.1 nor any of its corollaries.

Theorem 2.7. *If X is a regular space with a π -base whose elements have compact closure, then $|X| \leq 2^{wL(X)\psi(X)t(X)}$.*

Proof. Let $\kappa = wL(X)\psi(X)t(X)$ and let \mathcal{B} be the collection of all open sets with compact closure. As X is regular, we may find for any $p \in X$ a family of open sets \mathcal{U}_p such that $\{p\} = \bigcap \mathcal{U}_p = \bigcap \{clU : U \in \mathcal{U}_p\}$ and $|\mathcal{U}_p| \leq \kappa$. We may assume without loss of generality that each \mathcal{U}_p is closed under finite intersections. We construct by transfinite induction a non-decreasing chain of open sets $\{A_\alpha : \alpha < \kappa^+\}$ such that (1) $cl(A_\alpha) \leq 2^\kappa$ for all $\alpha < \kappa^+$, and (2) if $X \setminus cl(\bigcup \mathcal{W}) \neq \emptyset$ for $\mathcal{W} \in [\bigcup \{\mathcal{U}_p : p \in A_\alpha\}]^{\leq \kappa}$, then $A_{\alpha+1} \setminus cl(\bigcup \mathcal{W}) \neq \emptyset$.

For limit ordinals $\beta < \kappa^+$, let $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$. Then $|A_\beta| \leq 2^\kappa$. Using that $|Y| \leq d(Y)^{\psi(Y)t(Y)}$ for any regular space Y , it follows that $|cl(A_\beta)| \leq 2^\kappa$. For a successor ordinal $\beta + 1$, for every $\mathcal{W} \in [\bigcup \{\mathcal{U}_p : p \in A_\beta\}]^{\leq \kappa}$ such that $X \setminus cl(\bigcup \mathcal{W}) \neq \emptyset$, we chose $B_\mathcal{W} \in \mathcal{B}$ such that $cl(B_\mathcal{W}) \in X \setminus cl(\bigcup \mathcal{W})$. As each $cl(B_\mathcal{W})$ is compact, we see that $|cl(B_\mathcal{W})| \leq 2^{\psi(X)} \leq 2^\kappa$. We define

$$A_{\beta+1} = A_\beta \cup \bigcup \{B_\mathcal{W} : \mathcal{W} \in [\bigcup \{\mathcal{U}_p : p \in A_\beta\}]^{\leq \kappa} \text{ and } X \setminus cl(\bigcup \mathcal{W}) \neq \emptyset\}.$$

We again see that $|cl(A_{\beta+1})| \leq 2^\kappa$, using that $|Y| \leq d(Y)^{\psi(Y)t(Y)}$ for any regular space Y .

Let $F = \bigcup \{cl(A_\alpha) : \alpha < \kappa^+\}$ and note $|F| \leq 2^\kappa$. We show that $X = F$. By $t(X) \leq \kappa$, we have $F = cl(\bigcup \{A_\alpha : \alpha < \kappa^+\})$ and so F is regular-closed. Suppose that $X \neq F$ and choose a non-empty $B \in \mathcal{B}$ such that $clB \subseteq X \setminus F$. Observe that for any $p \in F$, we have $\bigcap \{clB \cap clU : U \in \mathcal{U}_p\} = \emptyset$. Now, the compactness of clB and the fact that \mathcal{U}_p is closed under finite intersections ensure the existence of some $U_p \in \mathcal{U}_p$ such that $U_p \cap clB = \emptyset$. As $\{U_p : p \in F\}$ is an open cover of F and $wL(X)$ is hereditary on regular-closed sets, there exists $\mathcal{W} \in \{U_p : p \in F\}^{\leq \kappa}$ such that $F \subseteq cl(\bigcup \mathcal{W})$. There exists

$\alpha < \kappa^+$ such that $\mathcal{W} \in [\bigcup\{\mathcal{U}_p : p \in A_\alpha\}]^{\leq \kappa}$. As $B \cap \bigcup \mathcal{W} = \emptyset$ it follows that $X \setminus cl(\bigcup \mathcal{W}) \neq \emptyset$. Thus, $B_{\mathcal{W}}$ is defined and we have $\emptyset \neq B_{\mathcal{W}} \subseteq A_{\alpha+1} \setminus cl(\bigcup \mathcal{W}) \subseteq F \setminus cl(\bigcup \mathcal{W}) = \emptyset$. As this is a contradiction we have $X = F$ and hence $|X| \leq 2^\kappa$. \square

The following corollary is immediate from Theorem 2.7.

Corollary 2.8. *If X is regular with a dense set of isolated points, then $|X| \leq 2^{wL(X)\psi(X)t(X)}$.*

In Question 4.1 in [3], it was asked whether $|X| \leq 2^{wL(X)\psi(X)t(X)}$ for a normal space X . Example 2.4 in that paper, described below, demonstrates that $|X| \leq 2^{wL(X)\psi(X)t(X)}$ does not hold for all regular spaces (nor, in fact, even for all zero-dimensional spaces). However, as we see in Theorem 2.7 above, if X is regular with the added requirement that X has a π -base whose elements have compact closure, then $|X| \leq 2^{wL(X)\psi(X)t(X)}$. This gives a partial answer to Question 4.1 in [3]. For the reader's benefit we give a description below of the space Y in Example 2.4 in [3].

Example 2.9 (Example 2.4 in [3]). Let κ be any uncountable cardinal, let \mathbb{Q} denote the rationals, and let A be any countable dense subset of the space of irrational numbers. Let Y be the set $(\mathbb{Q} \times \kappa) \cup A$ with the following topology. If $q \in \mathbb{Q}$ and $\alpha < \kappa$ then a neighborhood base at (q, α) is $\{U_n(q, \alpha) : n = 1, 2, \dots\}$ where $U_n(q, \alpha) = \{(r, \alpha) : r \in \mathbb{Q} \text{ and } |r - q| < 1/n\}$. If $a \in A$, $n \in \mathbb{N}$, and $F \in [\kappa]^{<\omega}$, let $V_{n,F}(a) = \{b \in A : |b - a| \leq 1/n\} \cup \{(q, \alpha) \in \mathbb{Q} \times \kappa : |q - \alpha| < 1/n \text{ and } \alpha \notin F\}$. Then $\{V_{n,F}(a) : n \in \mathbb{N} \text{ and } F \in [\kappa]^{<\omega}\}$ is a neighborhood base at a .

It can be shown that the space Y is a zero-dimensional Hausdorff space such that $wL(Y) = \aleph_0$, $t(Y) = \aleph_0$, and every subset of Y is G_δ (in particular, $\psi(Y) = \aleph_0$). Thus, if $\kappa > \mathfrak{c}$, we have $|Y| = \kappa > 2^{wL(Y)t(Y)\psi(Y)}$. We further observe that the family $\{U_n(q, \alpha) : q \in \mathbb{Q}, \alpha < \kappa, n \in \mathbb{N}\}$ is a π -base for Y consisting of countable sets. Thus, the condition in Theorem 2.7 above that X have a π -base whose elements have compact closure cannot be changed to the condition that X have a π -base whose elements are countable (or Lindelöf or c.c.c). \square

For a property \mathcal{P} of a space X , we say X is *locally* \mathcal{P} if every point in X has a neighborhood with property \mathcal{P} . We give a short proof below that if a Hausdorff space X has one of several local properties then $|X| \leq 2^{wL(X)\chi(X)}$. Note that in the case of the property locally normal we make the additional requirement that X be regular. The proof of Theorem 2.10 does not use Theorem 2.1, yet Theorem 2.10 follows from Corollary 2.5 in the case where X is Urysohn or quasiregular.

Theorem 2.10. *Let X be locally H -closed, locally Lindelöf, locally c.c.c., or regular and locally normal. Then $|X| \leq 2^{wL(X)\chi(X)}$.*

Proof. Let $\kappa = wL(X)\chi(X)$. One can find a cover \mathcal{U} of X such that U has a non-empty interior and is H -closed, Lindelöf or c.c.c. for each $U \in \mathcal{U}$. Moreover, if X is regular and locally normal, then the cover \mathcal{U} can consist of regular closed normal subspaces. Note that $|U| \leq 2^\kappa$ for all $U \in \mathcal{U}$ (the last case uses the fact that $wL(U) \leq wL(X)$). Choose a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq wL(X)$ and $\bigcup \mathcal{V}$ is dense in X . To finish, observe that $\bigcup \mathcal{V}$ has cardinality $\leq 2^{wL(X)\chi(X)}$ and apply the well-known inequality $|X| \leq d(X)^{\chi(X)}$, true for every Hausdorff space. \square

The last instance of the previous theorem is a proper strengthening of Bell-Ginsburg-Woods's result.

Corollary 2.11. *If X is locally compact then $|X| \leq 2^{wL(X)\psi(X)}$.*

Proof. Follows from Theorem 2.10 or Theorem 2.7 and the fact that X is of pointwise countable type; that is, X can be covered by compact subsets K such that $\chi(K, X) = \aleph_0$. (See 3.3.H in [11], for example). It is well-known that $\psi(X) = \chi(X)$ for every space of pointwise countable type. \square

In [1] Arhangel'skiĭ derived from his general theorem the proof that for any Lindelöf Hausdorff sequential space X with $\psi(X) \leq 2^{\aleph_0}$ the bound $|X| \leq 2^{\aleph_0}$ holds. We will show that a similar result is true for normal weakly Lindelöf spaces.

Theorem 2.12. *If X is a weakly Lindelöf normal sequential space satisfying $\chi(X) \leq 2^{\aleph_0}$, then $|X| \leq 2^{\aleph_0}$.*

Proof. For each $p \in X$ let \mathcal{U}_p be a base of open neighbourhoods at p satisfying $|\mathcal{U}_p| \leq 2^{\aleph_0}$. We will construct a non-decreasing collection $\{F_\alpha : \alpha < \omega_1\}$ of closed subsets of X in such a way that:

- 1) $|F_\alpha| \leq 2^{\aleph_0}$ for each α ;
- 2) if $X \setminus \overline{\bigcup \mathcal{V}} \neq \emptyset$ for a countable $\mathcal{V} \subset \bigcup \{\mathcal{U}_x : x \in F_\alpha\}$, then $F_{\alpha+1} \setminus \overline{\bigcup \mathcal{V}} \neq \emptyset$. Fix a choice function $\phi : \mathcal{P}(X) \rightarrow X$ and let $F_0 = \{\phi(\emptyset)\}$. Now, assume to have already constructed $\{F_\beta : \beta < \alpha\}$. If α is a limit ordinal, then put $F_\alpha = \overline{\bigcup \{F_\beta : \beta < \alpha\}}$. Condition 1) is fulfilled because in a sequential space we always have $|\overline{S}| \leq |S|^{\aleph_0}$. If $\alpha = \gamma + 1$, then F_α will be the closure of the set $F_\gamma \cup \{\phi(X \setminus \overline{\bigcup \mathcal{V}}) : \mathcal{V} \text{ a countable subset of } \bigcup \{\mathcal{U}_x : x \in F_\gamma\}\}$. Since a sequential space has countable tightness, the set $F = \bigcup \{F_\alpha : \alpha < \omega_1\}$ is closed and obviously we have $|F| \leq 2^{\aleph_0}$. So, to finish the proof it suffices to show that $X = F$.

Assume the contrary and fix a non-open set W such that $\overline{W} \cap F = \emptyset$. For every $x \in F$ choose an element $U_x \in \mathcal{U}_x$ satisfying $U_x \cap \overline{W} = \emptyset$. As X is weakly Lindelöf and normal, and F is closed, there is a countable collection $\mathcal{V} \subset \{U_x : x \in F\}$ such that $F \subset \overline{\bigcup \mathcal{V}}$. But, there exists some $\gamma < \omega_1$ satisfying $\mathcal{V} \subset \{U_x : x \in F_\gamma\}$ and this leads to a contradiction with condition 2). \square

With minor changes in the previous proof, we get a version similar to 2.7.

Theorem 2.13. *Let X be a weakly Lindelöf, normal, sequential space satisfying $\psi(X) \leq 2^{\aleph_0}$. If X has a π -base whose elements have compact closure, then $|X| \leq 2^{\aleph_0}$.*

Fundamental questions remain concerning which spaces X have cardinality bounded by $2^{wL(X)\chi(X)}$. As locally compact spaces are Tychonoff as well as Baire, we ask the following two general questions.

Question 2.14. *If X is Tychonoff, is $|X| \leq 2^{wL(X)\chi(X)}$?*

Question 2.15. *If X is a Baire space that is quasiregular or Urysohn, is $|X| \leq 2^{wL(X)\chi(X)}$?*

Recall a space X is *feebly compact* if for every countable open cover \mathcal{U} of X there exists $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $X = \bigcup_{V \in \mathcal{V}} clV$. It is well known that a feebly compact quasiregular space is Baire (see, for example, 7T(2) in [16]). We then ask the following question, which is a special case of Question 2.15.

Question 2.16. *If X is a feebly compact quasiregular space, is $|X| \leq 2^{wL(X)\chi(X)}$?*

Finally, as the properties pseudocompact and feebly compact are equivalent in the class of Tychonoff spaces, we also ask the following, which is a special case of both Questions 2.14 and 2.16.

Question 2.17. *If X is pseudocompact and Tychonoff, is $|X| \leq 2^{wL(X)\chi(X)}$?*

3. POWER HOMOGENEITY.

The fundamental closing-off argument of Theorem 2.1 can be used to develop a bound for the cardinality of a locally compact, power homogeneous space (Corollary 3.7). We begin with a bound on the θ -density of a power homogeneous space with a π -base consisting of “small” elements. Theorem 3.1 can be considered a companion theorem to Corollary 2.2.

For a space X , recall that the *pointwise compactness type* of X , denoted by $\text{pct}(X)$, is the least cardinal κ such that X can be covered by compact subsets K such that $\chi(K, X) \leq \kappa$. The cardinal invariant $\text{pct}(X)$ then generalizes the notion of pointwise countable type.

Theorem 3.1. *Let X be a power homogeneous space and suppose that X has a open π -base \mathcal{B} such that $|B| \leq 2^{wL(X)t(X)\text{pct}(X)}$ for all $B \in \mathcal{B}$. Then $d_\theta(X) \leq 2^{wL(X)t(X)\text{pct}(X)}$.*

Proof. The proof is a modification of the proof of Corollary 3.3(b) in [6]. Let $\kappa = wL(X)t(X)\text{pct}(X)$. As $t(X)\text{pct}(X) \leq \kappa$, by Lemma 3.1 in [6] there exists a non-empty compact set K and a set $A \in [X]^{\leq \kappa}$ such that $\chi(K, X) \leq \kappa$ and $K \subseteq \text{cl}A$. As $\pi_\chi(X) \leq t(X)\text{pct}(X) \leq \kappa$, a well-known generalization of Šapirovskiĭ's inequality $\pi_\chi(X) \leq t(X)$ for a compactum X , by Corollary 2.9 in [2] there exists a cover \mathcal{C} of X of compact sets of character at most κ such that each member of \mathcal{C} is contained in the closure of a subset of cardinality at most κ . By Theorem 2.1, there exists a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $X = \text{cl}_\theta(\bigcup \mathcal{C}')$ and $|\mathcal{C}'| \leq 2^\kappa$.

Now, for each $C \in \mathcal{C}'$ there exists $A_C \in [X]^{\leq \kappa}$ such that $C \subseteq \text{cl}(A_C)$. Let $\mathcal{A} = \{A_C : C \in \mathcal{C}'\}$ and note $|\mathcal{A}| \leq 2^\kappa$. For each $C \in \mathcal{C}$ we have that $C \subseteq \text{cl}(\bigcup \mathcal{A})$ and therefore $\bigcup \mathcal{C}' \subseteq \text{cl}(\bigcup \mathcal{A}) \subseteq \text{cl}_\theta(\bigcup \mathcal{A})$. It follows that $X = \text{cl}_\theta(\bigcup \mathcal{C}') \subseteq \text{cl}_\theta(\bigcup \mathcal{A})$, and so $d_\theta(X) \leq |\bigcup \mathcal{A}| \leq 2^\kappa \cdot \kappa = 2^\kappa$. \square

In Theorem 4.3 in [5] it was shown that $|X| \leq d_\theta(X)^{\pi_\chi(X)}$ for a Urysohn power homogeneous space X . As $d_\theta(X) = d(X)$ for a quasiregular space X and $|X| \leq d(X)^{\pi_\chi(X)}$ for any (Hausdorff) power homogeneous space X by a result of Ridderbos, Theorem 3.1 yields the following corollary.

Corollary 3.2. *Let X be a power homogeneous Hausdorff space that is either quasiregular or Urysohn. Suppose that X has a open π -base \mathcal{B} such that $|B| \leq 2^{wL(X)t(X)\text{pct}(X)}$ for all $B \in \mathcal{B}$. Then $|X| \leq 2^{wL(X)t(X)\text{pct}(X)}$.*

Proof. Let $\kappa = wL(X)t(X)\text{pct}(X)$. If X is quasiregular then $d_\theta(X) = d(X)$ and, by Theorem 3.4 in [18], $|X| \leq d(X)^{\pi_\chi(X)} = d_\theta(X)^{\pi_\chi(X)}$. If X is Urysohn, then $|X| \leq d_\theta(X)^{\pi_\chi(X)}$ by Theorem 4.3 [5]. In either case, by Theorem 3.1 above, we have $|X| \leq d_\theta(X)^{\pi_\chi(X)} \leq (2^\kappa)^{\pi_\chi(X)} \leq 2^\kappa$. \square

We move towards establishing a cardinality bound on any open subset of a power homogeneous space X (Theorem 3.5). That theorem can be regarded as an improvement of Corollary 3.11 in [7]. We first obtain some preliminary lemmas. Lemmas 3.3 and 3.4 are essentially

slight improvements of Corollary 3.3 and Theorem 3.4 from [18] and we use notation and proof techniques similar to that paper. For a space X , $x \in X$, a cardinal μ , and product space X^μ , we denote by \bar{x} the element of X^μ which is equal to x on all coordinates. The *diagonal* of X^μ is $\Delta(X, \mu) = \{\bar{x} : x \in X\}$. For a subset $C \subseteq \mu$, let $\pi_C : X^\mu \rightarrow X^C$ be the projection, and for $x \in X$ let x_C be the point $\pi_C(\bar{x}) \in X^C$. We also adopt the notation given before and after Theorem 3.1 in [18], which features heavily in the next proof.

Lemma 3.3. *Let X be a space such that X^μ is homogeneous for a cardinal μ . Fix $p \in X$ and suppose $\pi\chi(X) \leq \kappa$ for a cardinal κ . Let \mathcal{U} be a local π -base at p in X such that $|\mathcal{U}| \leq \kappa$, and suppose $\mu \geq \kappa$. Suppose further that $D \subseteq X$ and $Y \subseteq \text{cl}D$. Let $\pi : X^\mu \rightarrow X$ be any fixed projection. Then for all $x \in X$, there exists a homeomorphism $h_x : X^\mu \rightarrow X^\mu$ such that*

- (1) $h_x(\bar{p}) = \bar{x}$, and
- (2) if $B \in \mathcal{U}(\kappa)$ satisfies $\pi h_x \pi_\kappa^{-1}[B] \cap Y \neq \emptyset$, then there exists $e \in X^\mu$ and $d \in D$ satisfying
 - (a) $\pi h_x(e) = d \in D$ and $e \in \pi_\kappa^{-1}[B]$, and
 - (b) $h_x \pi_\kappa^{-1}(e_\kappa) \subseteq \pi^{-1}(d)$.

Proof. Fix $x \in X$. As X^μ is homogeneous, there exists a homeomorphism $h : X^\mu \rightarrow X^\mu$ such that $h(\bar{p}) = \bar{x}$. We inductively construct a sequence $\{A_n : n < \omega\} \subseteq [\mu]^{\leq \kappa}$ of subsets such that $|A_n| \leq \kappa$ for all $n < \omega$. Let $A_0 = \kappa$ and suppose A_n has been defined.

Let $C \in \mathcal{U}(A_n)$ and suppose that $\pi h \pi_{A_n}^{-1}[C] \cap Y \neq \emptyset$. Then $\pi h \pi_{A_n}^{-1}[C] \cap \text{cl}D \neq \emptyset$ and thus $\pi h \pi_{A_n}^{-1}[C] \cap D \neq \emptyset$. Let $d(C) \in \pi h \pi_{A_n}^{-1}[C] \cap D$, and select $e(C) \in \pi_{A_n}^{-1}[C]$ such that $d(C) = \pi h(e(C))$. If $\pi h \pi_{A_n}^{-1}[C] \cap Y = \emptyset$, select any $e(C) \in \pi_{A_n}^{-1}[C]$. Let $Z = \{e(C) : C \in \mathcal{U}(A_n)\}$ and note that $|Z| \leq |\mathcal{U}(A_n)| \leq \kappa$. Now apply Cor. 2.3 in [2] (also Prop. 2.2 in [18]) to obtain a set A_{n+1} such that $|A_{n+1}| \leq \kappa$ and $h \pi_{A_{n+1}}^{-1}((e(C))_{A_{n+1}}) \subseteq \pi^{-1} \pi h(e(C))$ for all $C \in \mathcal{U}(A_n)$. Note that if $C \in \mathcal{U}(A_n)$ is such that $\pi h \pi_{A_n}^{-1}[C] \cap Y \neq \emptyset$, then $h \pi_{A_{n+1}}^{-1}((e(C))_{A_{n+1}}) \subseteq \pi^{-1}(d(C))$.

Let $A = \bigcup_{n < \omega} A_n$. As $\kappa = A_0 \subseteq A$ and $|A| \leq \kappa$, we see that $|A| = \kappa$. There are two cases. First, if $\mu = \kappa$, then $A = \kappa$ and we have then proved above that h is the required homeomorphism satisfying the properties needed in the statement of the Lemma.

Otherwise, for the rest of the proof we suppose $\kappa < \mu$. In this case we use the coordinate change homeomorphism $g_{\kappa \rightarrow A} : X^\mu \rightarrow X^\mu$ (see notation following Thm 3.1 in [18]). Let $g = g_{\kappa \rightarrow A}$ and define $h_x = h \circ g$. Note $h_x(\bar{p}) = h(g(\bar{p})) = h(\bar{p}) = \bar{x}$, establishing (1) above.

We prove (2) holds for the homeomorphism h_x . Let $B \in \mathcal{U}(\kappa)$ and suppose that $\pi h_x \pi_\kappa^{-1}[B] \cap Y \neq \emptyset$. We see that $(g[\pi_\kappa^{-1}[B]])_A \in \mathcal{U}(A)$ by Lemma 3.2 in [18]. Let $B' = (g[\pi_\kappa^{-1}[B]])_A$. Also by Lemma 3.2 in [18], we have that

$$\begin{aligned} \emptyset \neq \pi h_x \pi_\kappa^{-1}[B] \cap Y &= \pi h[g[\pi_\kappa[B]]] \cap Y \\ &= \pi h \pi_A^{-1}[(g[\pi_\kappa^{-1}[B]])_A] \cap Y \\ &= \pi h \pi_A^{-1}[B'] \cap Y. \end{aligned}$$

Now, as noted at the end of the proof of Thm 3.1 in [18], $\mathcal{U}(A) = \bigcup_{n < \omega} \pi_{A \rightarrow A_{n+1}}^{-1} \mathcal{U}(A_{n+1})$. Thus there exists $n < \omega$ and $C \in \mathcal{U}(A_n)$ such that $B' = \pi_{A \rightarrow A_n}^{-1}[C]$. Now, for that n ,

$$\begin{aligned} \pi h \pi_{A_n}^{-1}[C] \cap Y &= \pi h \pi_A^{-1}[\pi_{A \rightarrow A_n}^{-1}[C]] \cap Y \\ &= \pi h \pi_A^{-1}[B'] \cap Y \\ &\neq \emptyset. \end{aligned}$$

As $C \in \mathcal{U}(A_n)$ and $\pi h \pi_{A_n}^{-1}[C] \cap Y \neq \emptyset$, we have $d(C)$ and $e(C)$ as defined in the second paragraph of this proof. Let $d = d(C)$ and $e' = e(C)$. Then $d \in \pi h \pi_{A_n}^{-1}[C] \cap D$, $e' \in \pi_{A_n}^{-1}[C]$, $d = \pi h(e')$, and $h \pi_{A_{n+1}}^{-1}(e'_{A_{n+1}}) \subseteq \pi^{-1}(d)$. But $\pi_A^{-1}(e'_A) \subseteq \pi_{A_{n+1}}^{-1}(e'_{A_{n+1}})$ and thus $h \pi_A^{-1}(e'_A) \subseteq h \pi_{A_{n+1}}^{-1}(e'_{A_{n+1}}) \subseteq \pi^{-1}(d)$. Furthermore, by Lemma 3.2 in [18],

$$\begin{aligned} d \in \pi h \pi_{A_n}^{-1}[C] \cap D &= \pi h \pi_A^{-1}[\pi_{A \rightarrow A_n}^{-1}[C]] \cap D = \pi h \pi_A^{-1}[B'] \cap D \\ &= \pi h \pi_A^{-1}[(g[\pi_\kappa^{-1}[B]])_A] = \pi h g[\pi_\kappa^{-1}[B]] \\ &= \pi h_x \pi_\kappa^{-1}[B], \end{aligned}$$

and

$$\begin{aligned} e' \in \pi_{A_n}^{-1}[C] &= \pi_A^{-1}[\pi_{A \rightarrow A_n}^{-1}[C]] = \pi_A^{-1}[B'] \\ &= \pi_A^{-1}[(g[\pi_\kappa^{-1}[B]])_A] = g[\pi_\kappa^{-1}[B]]. \end{aligned}$$

Thus there exists $e \in \pi_\kappa^{-1}[B]$ such that $e' = g(e)$. Thus, $d = \pi h(e') = \pi h g(e) = \pi h_x(e)$. Furthermore, as $\pi_A^{-1}(e'_A) = g \pi_\kappa^{-1}(e_\kappa)$ (again by Lemma 3.2 in [18]), we have that $h_x \pi_\kappa^{-1}(e_\kappa) = h g \pi_\kappa^{-1}(e_\kappa) = h \pi_A^{-1}(e'_A) \subseteq \pi^{-1}(d)$. This establishes the Lemma. \square

The next lemma is a strengthening of Theorem 3.4 in [18].

Lemma 3.4. *Let X be a power homogeneous space. If $D \subseteq X$ and U is an open set such that $U \subseteq \text{cl} D$, then $|U| \leq |D|^{\pi \chi(X)}$.*

Proof. Fix $p \in X$, let $\kappa = \pi\chi(X)$, and let μ be a cardinal such that X^μ is homogeneous. If $\mu \leq \kappa$, then X^κ is also homogeneous, so we can assume without loss of generality that $\mu \geq \kappa$. For all $\bar{x} \in \Delta(X, \mu)$ there exists a homeomorphism $h_x : X^\mu \rightarrow X^\mu$ satisfying (1) and (2) in Lemma 3.3, where $Y = U$. Now $\mathcal{U}(\kappa)$ is a local π -base at p_κ in X^κ of size at most κ . Fix $q \in D$.

We define $\phi : \Delta(U, \mu) \rightarrow D^{\mathcal{U}(\kappa)}$ as follows. Let $\bar{x} \in \Delta(U, \mu)$ and $B \in \mathcal{U}(\kappa)$. If $\pi h_x \pi_\kappa^{-1}[B] \cap U \neq \emptyset$, set $\phi(x)(B) = \pi h_x(e) = d \in D$, where e and d are as in 2(a) and 2(b) in Lemma 3.3. Otherwise, define $\phi(x)(B) = q$. Thus ϕ is well-defined.

We show ϕ is one-to-one. Suppose $\bar{x} \neq \bar{y} \in \Delta(U, \mu)$. There exist disjoint open sets V and W in X such that $x \in V$ and $y \in W$. So $x \in V \cap U$, $y \in W \cap U$, and

$$p_\kappa \in \pi_\kappa h_x^{-1} \pi^{-1}[V \cap U] \cap \pi_\kappa h_y^{-1} \pi^{-1}[W \cap U].$$

Note that the set on the right above is open in X^κ . There exists $B \in \mathcal{U}(\kappa)$ such that

$$B \subseteq \pi_\kappa h_x^{-1} \pi^{-1}[V \cap U] \cap \pi_\kappa h_y^{-1} \pi^{-1}[W \cap U],$$

from which it follows that $\pi h_x \pi_\kappa^{-1}[B] \cap U \neq \emptyset$ and $\pi h_y \pi_\kappa^{-1}[B] \cap U \neq \emptyset$.

Thus $\phi(x)(B) = \pi h_x(e) = d$ for e, d satisfying conditions 2(a) and 2(b) in Lemma 3.3. So $e \in \pi_\kappa^{-1}[B]$ and $e_\kappa \in B$. Since $B \subseteq \pi_\kappa h_x^{-1} \pi^{-1}[V \cap U]$, it follows that $\pi_\kappa^{-1}(e_\kappa) \cap h_x^{-1} \pi^{-1}[V \cap U] \neq \emptyset$. Applying h_x we have $h_x \pi_\kappa^{-1}(e_\kappa) \cap \pi^{-1}[V \cap U] \neq \emptyset$. Since $h_x \pi_\kappa^{-1}(e_\kappa) \subseteq \pi^{-1}(d)$, it follows that $d \in V \cap U$ and $\phi(x)(B) \in V \cap U$. Likewise, a similar argument shows $\phi(x)(B) \in W \cap U$.

As $V \cap W \neq \emptyset$, it follows that $\phi(x)(B) \neq \phi(y)(B)$ and therefore $\phi(x) \neq \phi(y)$. This shows π is an injection. It follows that $|U| = |\Delta(U, \mu)| \leq |D|^{\mathcal{U}(\kappa)} \leq |D|^\kappa$. \square

The following theorem represents a strengthening of Corollary 3.11 in [7].

Theorem 3.5. *If X be a power homogeneous space and $U \subseteq X$ is a non-empty open set, then $|U| \leq 2^{L(\overline{U})t(X)\text{pct}(X)}$.*

Proof. Let $\kappa = L(\overline{U})t(X)\text{pct}(X)$ and let $K = \overline{U}$. As $t(X)\text{pct}(X) \leq \kappa$, by Lemma 3.8 in [8] there exists a compact set of character at most κ contained in the closure of a set of size κ . Proposition 3.2 in [6], which is a slight variation of Corollary 2.9 in [2], guarantees there exists a family \mathcal{G} of compact sets of character at most κ and a family of subsets $\mathcal{H} = \{H_G : G \in \mathcal{G}\} \subseteq [X]^{\leq \kappa}$ such that a) $G \in cl(H_G)$ for all $G \in \mathcal{G}$, and b) $X = \bigcup \mathcal{G}$. Note that each $G \in \mathcal{G}$ is a G_κ -set in X . (Recall a G_κ -set is an intersection of κ -many open sets.)

Now, $K = \bigcup \{G \cap K : G \in \mathcal{G}\}$ and for all $G \in \mathcal{G}$, if $G \cap K \neq \emptyset$ then $G \cap K$ is a G_κ -set in the subspace K . Therefore, as K is compact, by Theorem 4 in Pytkeev [17], there exists $\mathcal{G}' \subseteq \mathcal{G}$ such that $|\mathcal{G}'| \leq 2^{t(K) \cdot \kappa} \leq 2^{t(X) \cdot \kappa} = 2^\kappa$ and $K \subseteq \bigcup \mathcal{G}'$.

Let $D = \bigcup \{H_G : G \in \mathcal{G}'\}$. Note that $|D| \leq |\mathcal{G}'| \cdot \kappa \leq 2^\kappa \cdot \kappa = 2^\kappa$. We have,

$$U \subseteq K \subseteq \bigcup \mathcal{G}' \subseteq \bigcup \{cl(H_G) : G \in \mathcal{G}'\} \subseteq cl\left(\bigcup \{H_G : G \in \mathcal{G}'\}\right) = clD.$$

By Lemma 3.4 and the fact that $\pi\chi(X) \leq t(X)\text{pct}(X)$ for any space X , it follows that $|U| \leq |D|^{\pi\chi(X)} \leq (2^\kappa)^{\pi\chi(X)} \leq (2^\kappa)^\kappa = 2^\kappa$. \square

We now tie our results in with Corollary 3.2.

Corollary 3.6. *Let X be a power homogeneous space that is either quasiregular or Urysohn. If X has a π -base \mathcal{B} such that clB is Lindelöf for all $B \in \mathcal{B}$ then $|X| \leq 2^{wL(X)t(X)\text{pct}(X)}$.*

Proof. Let $\kappa = wL(X)t(X)\text{pct}(X)$. By Theorem 3.5 it follows that $|B| \leq 2^{t(X)\text{pct}(X)} \leq 2^\kappa$ for all $B \in \mathcal{B}$. By Corollary 3.2, $|X| \leq 2^\kappa$. \square

Corollary 3.7. *Let X be a locally compact power homogeneous space. Then $|X| \leq 2^{wL(X)t(X)}$.*

Proof. Note that X has a π -base (in fact, a base) \mathcal{B} of open sets such that clB is compact for all $B \in \mathcal{B}$. Also X is of pointwise countable type, i.e. $\text{pct}(X) = \aleph_0$. By Corollary 3.6, $|X| \leq 2^{wL(X)t(X)\text{pct}(X)} = 2^{wL(X)t(X)}$. \square

We conclude with a question analogous to Question 2.14.

Question 3.8. *If X is power homogeneous and Tychonoff, is $|X| \leq 2^{wL(X)t(X)\text{pct}(X)}$?*

REFERENCES

- [1] A. V. Arhangel'ski, *On the cardinality of bicomacta satisfying the first axiom of countability*, Soviet Math. Dokl. 10 (1969) 951–955.
- [2] A. V. Arhangel'skiĭ, J. van Mill, and G. J. Ridderbos, *A new bound on the cardinality of power homogeneous compacta*, Houston J. Math. 33 (2007), no. 3, 781–793.
- [3] M. Bell, J. Ginsburg, G. Woods, *Cardinal inequalities for topological spaces involving the weak Lindelöf number*, Pacific Journal of Mathematics 79 (1978), no. 1, 37–45.
- [4] A. Bella, F. Cammaroto, *On the cardinality of Urysohn spaces*, Canad. Math. Bull. 31 (1988) 153–158.
- [5] N. A. Carlson, *Non-regular power homogeneous spaces*, Topology Appl. 154 (2007), no. 2, 302–308.

- [6] N. A. Carlson, *The weak Lindelöf degree and homogeneity*, Topology Appl. 160 (2013), no. 3, 508–512.
- [7] N. A. Carlson, G.J. Ridderbos, *On several cardinality bounds on power homogeneous spaces*, Houston Journal of Mathematics, 38 (2012), no. 1, 311–332.
- [8] N. Carlson, J.R. Porter, G.J. Ridderbos, *On Cardinality Bounds for Homogeneous Spaces and the G_κ -modification of a Space*, Topology Appl. 159, (2012), no. 13, 2932–2941.
- [9] R. de la Vega, *A new bound on the cardinality of homogeneous compacta*, Topology Appl. 153 (2006), 2118–2123.
- [10] A. Dow, J.R. Porter, *Cardinalities of H -closed spaces*, Topology Proc. 7 (1982), no. 1, 27–50.
- [11] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, second ed., 1989.
- [12] I. Gotchev, *Cardinalities of weakly Lindelöf spaces with regular G_κ diagonals*, preprint.
- [13] A. Hajnal, I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. 29 (1967) 343–356.
- [14] R.E. Hodel, *Arhangel'skii's solution to Alexandroff's problem: A survey*, Top. and its Appl. 153 (2006) 2199–2217.
- [15] I. Juhász, *Cardinal functions in topology—ten years later*, second ed., Mathematical Centre Tracts, vol. 123, Mathematisch Centrum, Amsterdam, 1980.
- [16] J. R. Porter, G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer, Berlin, 1988.
- [17] E. G. Pytkeev, *About the G_λ -topology and the power of some families of subsets on compacta*, Colloq. Math. Soc. Janos Bolyai 41 (1985), 517–522.
- [18] G. J. Ridderbos, *On the cardinality of power homogeneous Hausdorff spaces*, Fund. Math. 192 (2006), 255–266.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CATANIA, VIALE A. DORIA
6, 95125 CATANIA, ITALY

E-mail address: bella@dmi.unict.it

DEPARTMENT OF MATHEMATICS, CALIFORNIA LUTHERAN UNIVERSITY, 60 W.
OLSEN RD, MC 3750, THOUSAND OAKS, CA 91360 USA

E-mail address: ncarlson@callutheran.edu